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He's variational iteration method for solving nonlinear mixed Volterra–Fredholm integral equations

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ABSTRACT

In this paper He's variational iteration method is used to give the approximate solution of nonlinear mixed Volterra–Fredholm integral equations. The method constructs a convergent sequence of functions, which approximates the exact solution with few iterations. Numerical results and a comparison with the exact solution are given, which reveal its efficiency.

1. Introduction

The mixed Volterra–Fredholm integral equations arise in the theory of parabolic boundary value problems, the mathematical modeling of the spatio-temporal development of an epidemic, and various physical and biological problems. A discussion of the formulation of these models is given in [1–3] and the references therein.

The nonlinear mixed Volterra–Fredholm integral equation is given in [3] as

$$u(x, y) = f(x, y) + \int_0^y \int_\Omega G(x, y, s, t, u(s, t))\mathrm{d}s\mathrm{d}t, \quad (x, y) \in [0, y] \times \Omega, \tag{1}$$

where $u(x, y)$ is an unknown function, the functions $f(x, y)$ and $G(x, y, s, t, u)$ are analytic on $D = \Omega \times [0, T]$ and where Ω is a closed subset of $(\mathbb{R}^n,\ n = 1, 2, 3)$. The existence and uniqueness results for Eq. (1) may be found in [1,4–6]. However, few numerical methods for Eq. (1) are known in the literature [3]. For the linear case, the time collocation method was introduced in [6] and the projection method was presented in [7,8]. In [9] the results of [6] have been extended to nonlinear Volterra–Hammerstein integral equations. In [10,3], a technique based on the Adomian decomposition method was used for the solution of Eq. (1). A new method for solving Eq. (1) by means of the Legendre wavelets method is introduced in [11].

Another type of mixed Volterra–Fredholm integral equation that we are going to solve approximately by the variational iteration method is given as

$$y(x) = f(x) + \lambda_1 \int_0^x K_1(x, t)F(y(t))\mathrm{d}t + \lambda_2 \int_0^1 K_2(x, t)G(y(t))\mathrm{d}t, \quad 0 \le x, t \le 1, \tag{2}$$

where $f(x)$ and the kernels $K_1(x, t)$ and $K_2(x, t)$ are assumed to be in $L^2(\mathbb{R})$ on the interval $0 \le x, t \le 1$.

The variational iteration method is a new method for solving nonlinear problems and was introduced by a Chinese mathematician, He [12–15]. In [15] He modified the general Lagrange multiplier method [16] and constructed an iterative

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sequence of functions which converges to the exact solution. In most linear problems, on determining the exact Lagrange multiplier, the approximate solution turns into the exact solution and is available with just one iteration.

The variational iteration method is used to address several problems, most of them well-known differential equations, which arise in several fields of physics and engineering. A criterion for convergence of the variational iteration method is introduced in [17] by means of the Banach fixed point theorem.

In this paper we use the variational iteration method for solving the mixed Volterra–Fredholm equations. With this aim we differentiate Eq. (1) in the y direction: hence, Eq. (1) turns into an integral–differential equation, and then we use the variational iteration method in the same direction in order to find the approximate solution. For Eq. (2) we substitute the integration of an unknown function; then we use the variational iteration method.

The article is organized as follows. In Section 2, we describe the basic formulation of the variational iteration method required for our subsequent development. Section 3 is devoted to the solution of Eqs. (1) and (2) by using the variational iteration method. In Section 4, we report our numerical findings and demonstrate the accuracy of the proposed scheme by considering numerical examples.

2. Variational iteration method

The variational iteration method is used in [18] to solve the Fokker–Planck equation. This technique computes the exact solution of equations using the initial condition only. It is also important to note that the present method does not require discretization of the equation. Therefore, it is not affected by computation round-off errors and one is not faced with the necessity of large computer memory and time. Furthermore, using this idea we do not need to solve any linear or nonlinear system of equations. The authors of [19] applied He's variational iteration method for finding the minimum of a functional over the specified domain. Using this technique the solution of the problem is provided in a closed form while the mesh point techniques provide the approximation at mesh points only. In [20] the variational iteration method is employed to solve the time-dependent reaction–diffusion equation which has special importance in engineering and sciences and constitutes a good model for many systems in various fields. The Lane–Emden equation is solved in [21] by the variational iteration method. The results obtained show the efficiency and applicability of this procedure for solving it. This method is employed in [22] to solve the Klein–Gordon equation which is the relativistic version of the Schrödinger equation, which is used to describe spinless particles. Application of He's variational iteration technique to an inverse parabolic problem is described in [23]. The variational iteration method and Adomian decomposition method are used and compared in [24], for solving a biological population equation. The main advantage of the two methods over the mesh points methods [25] is the fact that they do not require discretization of the variables. Furthermore, the variational iteration method overcomes the difficulty arising in calculating the Adomian polynomials which is an important advantage over the Adomian decomposition method [36,37]. The authors of [26] used the variational iteration method to solve a system of two nonlinear integro-differential equations which arises in biology, describing biological species living together. The variational iteration method is applied in [38] to find solution of an inverse problem for the semi-linear parabolic partial differential equation. Authors of [39] modified the variational iteration method to solve a system of differential equations. The variation iteration procedure is employed in [40] to solve the wave equation subject to an integral conservation condition. The generalized pantograph equation [41] is investigated in [42] and the He's variation iteration technique is employed to solve it. The parabolic integro-differential equations arising in heat conduction in materials with memory is studied in [43] and the variational iteration method is used to find the solution of the problem. Authors of [44] used this method to solve several classes of variational problems. The interested reader can see [27–33] for some other applications of the method.

Consider the following general nonlinear problem:

$$L(u(t)) + N(u(t)) = g(t), \tag{3}$$

where L is a linear operator, N is a nonlinear operator, and $g(t)$ is a known analytical function. The variational iteration method constructs an iterative sequence called the correction functional as

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(L(u_n(s)) + N(\widetilde{u}_n(s)) - g(s))\mathrm{d}s,$$

where λ is the general Lagrange multiplier [16] which can be identified optimally via the variational theory [16,34], $\widetilde{u}_n(s)$ is considered as the restricted variation [35], i.e. $\delta\widetilde{u}_n = 0$, and the index n denotes the nth iteration.

3. Solution of the mixed Volterra–Fredholm integral equations

Consider the mixed integral equations given in Eqs. (1) and (2):

$$u(x, y) = f(x, y) + \int_0^y \int_\Omega G(x, y, s, t, u(s, t))\mathrm{d}s\mathrm{d}t, \quad (x, y) \in [0, y] \times \Omega, \tag{4}$$

with $\Omega = [0, 1]$.

$$y(x) = f(x) + \lambda_1 \int_0^x K_1(x, t)F(y(t))\mathrm{d}t + \lambda_2 \int_0^1 K_2(x, t)G(y(t))\mathrm{d}t, \quad 0 \le x, t \le 1. \tag{5}$$

For Eq. (1) first we take the partial derivative with respect to y. We have

$$\frac{\partial u}{\partial y}-\frac{\partial f}{\partial y}-\int_0^1 G(x,y,s,y,u(s,y))\mathrm{d}s-\int_0^y\int_0^1\frac{\partial G}{\partial y}\mathrm{d}s\mathrm{d}t=0.$$

Consider

$$-\int_0^1 G(x,y,s,y,u(s,y))\mathrm{d}s-\int_0^y\int_0^1\frac{\partial G}{\partial y}\mathrm{d}s\mathrm{d}t,$$

as a restricted variation; we use the variational iteration method in direction y. Then we have the following iteration sequence:

$$u_{n+1}(x,y)=u_n(x,y)+\int_0^y\lambda\left[\frac{\partial u_n}{\partial \tau}(x,\tau)-\frac{\partial f}{\partial \tau}(x,\tau)-\int_0^1 G(x,\tau,s,\tau,u_n(s,\tau))\mathrm{d}s-\int_0^\tau\int_0^1\frac{\partial G}{\partial \tau}\mathrm{d}s\mathrm{d}t\right]\mathrm{d}\tau.$$

Taking the variation with respect to the independent variable u_n and noticing that $\delta u_n(0)=0$, we get

$$\delta u_{n+1}=\delta u_n+\lambda\delta u_n|_{\tau=y}-\int_0^y\lambda'\delta u_n\mathrm{d}\tau=0.$$

Then we apply the following stationary conditions:

$$1+\lambda(\tau)|_{\tau=y}=0,\qquad \lambda'(\tau)|_{\tau=y}=0.$$

The general Lagrange multiplier, therefore, can be readily identified:

$$\lambda=-1,$$

and as a result, we obtain the following iteration formula:

$$u_{n+1}(x,y)=u_n(x,y)-\int_0^y\left[\frac{\partial u_n}{\partial \tau}(x,\tau)-\frac{\partial f}{\partial \tau}(x,\tau)-\int_0^1 G(x,\tau,s,\tau,u_n(s,\tau))\mathrm{d}s-\int_0^\tau\int_0^1\frac{\partial G}{\partial \tau}\mathrm{d}s\mathrm{d}t\right]\mathrm{d}\tau.$$

For the mixed integral equation (2), let $w(x)$ be a function such that $w'(x)=y(x)$, noting that $y(x)$ is continuous. Then we have

$$w'(x)=f(x)+\lambda_1\int_0^x K_1(x,t)F(w'(t))\mathrm{d}t+\lambda_2\int_0^1 K_2(x,t)G(w'(t))\mathrm{d}t.$$

Consider

$$\lambda_1\int_0^x K_1(x,t)F(w'(t))\mathrm{d}t+\lambda_2\int_0^1 K_2(x,t)G(w'(t))\mathrm{d}t,$$

as a restricted variation; we have the iteration sequence

$$w_{n+1}=w_n+\int_0^x\lambda\left[w_n'(s)-\lambda_1\int_0^s K_1(s,t)F(w_n'(t))\mathrm{d}t-\lambda_2\int_0^1 K_2(s,t)G(w_n'(t))\mathrm{d}t-f(s)\right]\mathrm{d}s.$$

Taking the variation with respect to the independent variable w_n and noticing that $\delta w_n(0)=0$, we get

$$\delta w_{n+1}=\delta w_n+\lambda(s)\delta w_n|_{s=x}-\int_0^x\lambda'(s)\delta w_n\mathrm{d}s=0.$$

Now we apply the following stationary conditions:

$$1+\lambda(s)|_{s=x}=0,\qquad \lambda'(s)|_{s=x}=0.$$

The general Lagrange multiplier, therefore, can be readily identified:

$$\lambda=-1,$$

and, as a result, we obtain the following iteration formula:

$$w_{n+1}=w_n-\int_0^x\left[w_n'(s)-\lambda_1\int_0^s K_1(s,t)F(w_n'(t))\mathrm{d}t-\lambda_2\int_0^1 K_2(s,t)G(w_n'(t))\mathrm{d}t-f(s)\right]\mathrm{d}s.$$

4. Illustrative examples

In this section we applied the method presented in this paper to two examples to show the efficiency of the approach.

4.1. Example 1

Consider the nonlinear mixed integral equation

$$u(x,y)=xy-\mathrm{e}^y+y+1+\int_0^y\int_0^1 t\mathrm{e}^{u(s,t)}\mathrm{d}s\mathrm{d}t,\quad 0\le y\le 1.\tag{6}$$

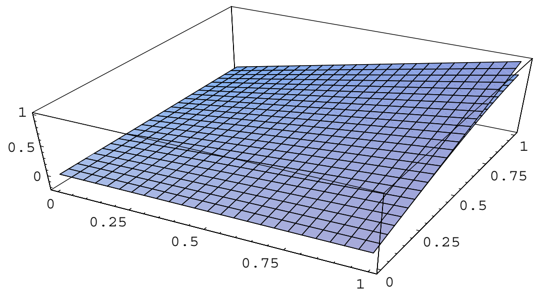

Fig. 1. Exact and approximate solutions for Example 1.

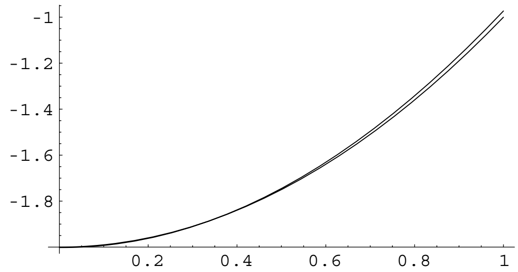

Fig. 2. Exact and approximate solutions for Example 2.

The exact solution is $u(x, y) = xy$. We applied the method presented in this paper and solved Eq. (6). The iteration formula for this example is

$$u_{n+1}(x, y) = u_n(x, y) - \int_0^y \left[\frac{\partial u_n}{\partial \tau}(x, \tau) - \tau \int_0^1 e^{u_n(s,\tau)} ds + e^{\tau} - x - 1 \right] d\tau. \tag{7}$$

Let $u_0(x, y) = 0$. Applying the iteration formula (7), the eighth approximate solution is $u_8(x, y)$. The exact solution and $u_8(x, y)$ are plotted in Fig. 1.

4.2. Example 2

Consider the nonlinear Volterra–Fredholm integral equation given in [11] by

$$y(x) = -\frac{1}{30}x^6 + \frac{1}{3}x^4 - x^2 + \frac{5}{3}x - \frac{5}{4} + \int_0^x (x - t)(y^2(t))dt + \int_0^1 (x + t)y(t)dt, \tag{8}$$

with $y(x) = x^2 - 2$, as the exact solution. Let $w'(x) = y(x)$. Then the iteration formula is

$$w_{n+1}(x) = w_n(x) - \int_0^x \left[w_n'(s) - \int_0^s (s - t)(w_n'(t))^2 dt - \int_0^1 (s + t)w_n'(t)dt + \frac{1}{30}s^6 - \frac{1}{3}s^4 + s^2 - \frac{5}{3}s + \frac{5}{4} \right] ds. \tag{9}$$

Let $w_0(x) = 1$. Applying the iteration formula (9), the sixth approximate solution is $w_6(x)$. The exact solution and $y_6(x)$ $(=w_6'(x))$ are plotted in Fig. 2.

Finally we would like to mention that the technique developed in this paper can be employed to solve the second Painleve equation investigated in [45].

5. Conclusion

In this paper the variational iteration method is used to solve the mixed Volterra–Fredholm integral equations. We described the method, used it on two test problems, and compared the results with their exact solutions in order to demonstrate the validity and applicability of the method. Moreover, only a small number of iterations are needed to obtain a satisfactory result. The given numerical examples support this claim.

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